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UNIVERSAL REPRESENTATIONS OF BRAID AND BRAID-PERMUTATION GROUPS

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Abstract. Drinfel'd used associators to construct families of universal representations of braid groups. We consider semi-associators (i.e., we drop the pentagonal axiom and impose a normalization in degree one). We show that the process may be reversed, to obtain semi-associators from universal representations of 3–braids. We view braid groups as subgroups of braid-permutation groups. We construct a family of universal representations of braid-permutation groups, without using associators. All representations in the family are faithful, defined over $\mathbb{Q}$ by simple explicit formulae. We show that they give universal Vassiliev–type invariants for braid-permutation groups.

1. Introduction

1.1. In the foundational paper [7], Drinfel'd introduced and proved the existence of associators, that is, formal series in two noncommutative variables with coefficients in a characteristic zero field $\mathbb{K}$, satifying certain axioms. These objects constitute the core structure leading to many important results. They play a key role in the quantization of universal enveloping algebras. There is a deep connection between associators and the absolute Galois group. They appear in an essential way in the construction of universal finite type invariants in low-dimensional topology. See [7], and also Birman's survey [3] and the monograph [11] by Kassel.

1.2. One also finds in [7] a bridge between associators and *universal representations* of Artin braid groups into braid algebras. These algebras (defined over $\mathbb{Z}$) are semidirect products of symmetric group algebras $\mathbb{K}[\Sigma_n]$ and infinitesimal Artin algebras $\mathcal{A}_n$ (alias complete algebras of horizontal chord diagrams on n strings). Given an associator, Drinfel'd constructs a family of so-called universal representations of the braid groups into the corresponding braid algebras, satisfying certain natural properties. See sections 2 and 3 for details.

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We consider in this note semi-associators, i.e., associators not required to verify the pentagonal axiom, which are still normalized in degree one; see Definition 3.2.

Theorem 1.1. *There is a natural bijection between semi-associators and universal representations of* 3*-braids.*

In the above bijection, the universal 3–representation coming from an associator coincides with the one constructed in [7]; see Theorem 4.3 and Remark 4.8. In other words, the Drinfel'd approach may also be used to obtain semi-associators from universal representations of 3–braids.

1.3. The Drinfel'd representations from [7] are faithful, as follows from work by Kohno [13]. There is however a practical inconvenient. Known explicit formulae for $\mathbb{C}$–associators involve complicated multiple zeta values, see for instance [11, Ch. XIX]. Over $\mathbb{Q}$, there is no example of explicitly described associator, to our best knowledge. To remedy this, we view Artin braid groups $\mathcal{B}_n$ inside the *welded braid groups* $\mathcal{BP}_n$ introduced and studied by Fenn–Rimanyi–Rourke [9] (also known as *braid-permutation groups*).

We propose analogs of braid algebras (also defined over $\mathbb{Z}$), in this enlarged context, namely *oriented braid algebras*; see Definitions 5.2 and 5.1. These are semidirect product algebras, $\mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]$, where the *oriented Artin algebra* $\mathcal{O}_n$ is a cousin of $\mathcal{A}_n$, obtained from *oriented* horizontal chord diagrams. There is also a natural notion of universal family of representations, of either welded or Artin braid groups into oriented braid algebras; this notion is defined by conditions (1) and (3) from Theorem 6.1. Our next goal is to point out similarities between universal representations of braid and braid-permutation groups, into braid algebras (respectively oriented braid algebras), emphasizing the fact that the latter exhibit simpler qualitative properties. We prove in Theorem 6.1 the following.

Theorem 1.2. *There is a universal family of faithful representations, for both welded and Artin braid groups, into oriented braid algebras. These representations are defined by explicit formulae, over* $\mathbb{Q}$.

The key point in the construction of associators and Drinfel'd representations is the analysis of the corresponding KZ–monodromy. The required flatness of KZ connection forms is intimately related to the 1–*formality* property of pure braid groups, in the sense of D. Sullivan [24]; see Section 5 for details.

The analogs of pure braid groups, in the context of welded braids, are McCool's groups from [15]. Our key step in proving Theorem 1.2 is to show that the McCool groups are 1–formal, and to compute their rational Lie algebras associated to the lower central series filtration. This is done in Theorem 5.4. See also Cohen–Pakianathan–Vershinin–Wu [6] for related results, in particular for the determination of the integral associated graded Lie algebra of upper-triangular

McCool groups. New information on upper-triangular McCool groups (which are proper subgroups of McCool groups) may be found in Remark 5.6.

1.4. It is well-known that Drinfel'd representations have the following geometric interpretation, see for instance [21, Section 1]. Consider on rational group rings of Artin braid groups the multiplicative *Vassiliev filtration*, obtained by resolving singularities of singular braids. On braid algebras, there is the natural multiplicative filtration coming from the complete filtration of infinitesimal Artin algebras $\mathcal{A}_n$. Drinfel'd representations are *universal finite type invariants* for the corresponding Artin braid groups. This means that their canonical extensions to group rings respect the above filtrations, and induce a multiplicative isomorphism at the associated graded level.

Our explicit representations of welded braid groups into oriented braid algebras from Theorem 1.2 have the same geometric flavour. On rational group rings of welded braids, we consider the multiplicative filtration obtained by resolving singularities (welds), as explained in §6.2. The complete filtration of oriented Artin algebras $\mathcal{O}_n$ naturally induces another multiplicative filtration, on oriented braid algebras. The result below is proved in Section 6.

Theorem 1.3. *The canonical extensions to group rings of the representations of welded braids into oriented braid algebras, constructed in Theorem* 1.2*, respect the filtrations described above, and induce a multiplicative isomorphism at the associated graded level.*

It is straightforward to check that the inclusion of group rings, $\mathbb{Q}[\mathcal{B}_n] \hookrightarrow \mathbb{Q}[\mathcal{BP}_n]$, respects the abovementioned filtrations. Consequently, the universal representations from Theorem 1.2, $\mathbb{Q}[\mathcal{BP}_n] \to \mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]$, give finite type invariants for braids, by restriction to $\mathbb{Q}[\mathcal{B}_n]$. For $n \leq 3$, it turns out that all finite type invariants for braids arise in this way. See Section 6 for more details, including a quantitative comparison between the associated graded algebras, $\mathcal{A}_n^*$ and $\mathcal{O}_n^*$.

2. Dramatis personae

Let us present the objects that inspired our study: braid groups and braid algebras.

2.1. The *braid* and *pure braid* groups were defined and studied by E. Artin [1]. The geometric braid group on n strings is isomorphic to the group $\mathcal{B}_n$ generated by $\sigma_1, \ldots, \sigma_{n-1}$, with defining relations

$$\left\{ \begin{array}{lll} \sigma_i\sigma_j = \sigma_j\sigma_i\,, & \text{for} & 2 \leq |i-j|\,; \\ \sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i\,, & \text{for} & 1 \leq i \leq n-2 \end{array} \right. \tag{1}$$

Using the natural morphism onto the symmetric group Σ_n, $\sigma_i \mapsto s_i := (i, i+1)$, Artin identified its kernel with the subgroup of *pure braids*:

$$1 \to \mathcal{PB}_n \to \mathcal{B}_n \to \Sigma_n \to 1 \,. \tag{2}$$

As a set of generators for $\mathcal{PB}_n$ we choose the elements

$$\alpha_{ji} = \sigma_{i-1}\sigma_{i-2}\cdots\sigma_{j+1}\sigma_j^2\sigma_{j+1}^{-1}\cdots\sigma_{i-2}^{-1}\sigma_{i-1}^{-1}\,, \quad 1 \le j < i \le n\,.$$

We also have a canonical embedding

$$\mathcal{B}_{n-1} \hookrightarrow \mathcal{B}_n\,, \text{ given by } \sigma_i \mapsto \sigma_i\,, i = 1, \ldots, n-2\,, \tag{3}$$

and an obvious stability property, expressed by the commuting diagram

$$\begin{array}{ccccccccc} 1 & \to & \mathcal{PB}_{n-1} & \to & \mathcal{B}_{n-1} & \to & \Sigma_{n-1} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathcal{PB}_n & \to & \mathcal{B}_n & \to & \Sigma_n & \to & 1 \end{array}$$

See also [16] for complete proofs.

2.2. To define the corresponding braid algebras, we need *complete topological Hopf algebras* and *complete topological Lie algebras*; see [23] for details. Consider the tensor algebra, i.e., the free associative algebra with coefficients in a field $\mathbb{K}$ of characteristic 0 , $\mathbb{T}_{\mathbb{K}}\langle A_1, \ldots, A_n\rangle$, where $\deg A_i = 1$ and the comultiplication is defined by $\Delta A_i = A_i \otimes 1 + 1 \otimes A_i$. We denote by $\mathbb{K}\langle\langle A_1, \ldots, A_n\rangle\rangle$ the completion of this Hopf algebra, with respect to the degree filtration. Its primitive part is the complete free Lie algebra, $\widehat{\mathbb{L}}\langle A_1, \ldots, A_n\rangle = \operatorname{Prim} \mathbb{K}\langle\langle A_1, \ldots, A_n\rangle\rangle$. We denote by $\widehat{\mathbb{T}}^{>k}\langle A_1, \ldots, A_n\rangle$ and $\widehat{\mathbb{L}}^{>k}\langle A_1, \ldots, A_n\rangle$ the corresponding complete filtrations. We denote the congruence modulo these ideals by $\equiv_{k+1}$; for instance, $f \equiv_2 g$ means that f and g have the same linear part and the same constant term, if $f, g \in \mathbb{K}\langle\langle A_1, \ldots, A_n\rangle\rangle$. Similar considerations apply to arbitrary complete Hopf and Lie algebras.

Kohno [12] and Drinfel'd [7] introduced infinitesimal versions of (pure) braid groups.

Definition 2.1 ([12])**.** The *infinitesimal Artin Hopf algebra* is the complete Hopf algebra given by the presentation:

$$\mathcal{A}_n = \mathbb{K}\langle\langle t_{ij} = t_{ji}, 1 \le i \ne j \le n \mid [t_{ij}, t_{ik} + t_{jk}] = 0, [t_{ij}, t_{kl}] = 0 \text{ if } \{i,j\} \cap \{k,l\} = \emptyset\rangle\rangle,$$

where $[u, v] := uv - vu$ denotes the algebra commutator. The *infinitesimal Artin Lie algebra* is the complete Lie algebra given by the presentation:

$$\mathcal{P}_n = \widehat{\mathbb{L}}\langle t_{ij} = t_{ji}, 1 \le i \ne j \le n \mid [t_{ij}, t_{ik} + t_{jk}] = 0, [t_{ij}, t_{kl}] = 0 \text{ if } \{i,j\} \cap \{k,l\} = \emptyset\rangle$$

These two algebras determine each other: $\mathcal{A}_n = \mathcal{U}\mathcal{P}_n$ and $\mathcal{P}_n = \mathrm{Prim}(\mathcal{A}_n)$.

There is a natural left action of the symmetric group Σ_n on the above algebras, defined by $\pi(t_{ij}) = t_{\pi(i)\pi(j)}$. We shall use the exponential notation $\pi(\Phi) := {}^{\pi}\Phi$. For instance, if $\pi = ijk := \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \in \Sigma_3$, we denote $\pi(\Phi)$ by ${}^{ijk}\Phi$.

Definition 2.2 ([7])**.** The *braid algebra* $\mathcal{A}_n \rtimes \mathbb{K}[\Sigma_n]$ is the semidirect algebra product, that is, $\mathcal{A}_n \otimes \mathbb{K}[\Sigma_n]$, with twisted multiplication given by $(a \otimes x) \cdot (b \otimes y) = a \cdot {}^{x}b \otimes xy$.

The algebra $\mathcal{A}_n \rtimes \mathbb{K}[\Sigma_n]$ contains as a multiplicative subgroup the semidirect group product $\exp \mathcal{P}_n \rtimes \Sigma_n$. We thus have split exact sequences of groups,

$$1 \to \exp \mathcal{P}_n \to \exp \mathcal{P}_n \rtimes \Sigma_n \to \Sigma_n \to 1\,,$$

together with compatible canonical embeddings,

$$\exp \mathcal{P}_{n-1} \rtimes \Sigma_{n-1} \hookrightarrow \exp \mathcal{P}_n \rtimes \Sigma_n\,.$$

2.3. We will start by considering families of representations,

$$\rho_n : \mathcal{B}_n \to \mathcal{A}_n \rtimes \mathbb{K}[\Sigma_n]\,,$$

satisfying the following four natural properties, extracted from the work of Drinfel'd [7].

- **Exponential type.** The representation ρ_n factorizes through the exponential subgroup:

$$(\mathbf{E}) \qquad \begin{tikzcd} \mathcal{B}_n \arrow[r, "\rho_n"] \arrow[dr, "\rho_n"'] & \exp \mathcal{P}_n \rtimes \Sigma_n \arrow[d] \\ & \mathcal{A}_n \rtimes \mathbb{K}[\Sigma_n] \end{tikzcd}$$

- **Symmetry.** The diagram below commutes:

$$(\mathbf{\Sigma}) \qquad \begin{tikzcd} \mathcal{B}_n \arrow[r, "\rho_n"] \arrow[d] & \exp \mathcal{P}_n \rtimes \Sigma_n \arrow[d] \\ \Sigma_n \arrow[r, equal] & \Sigma_n \end{tikzcd}$$

- **Stability.** One has commutative diagrams

$$(\mathbf{S})\qquad \begin{tikzcd} \mathcal{B}_{n-1} \arrow[r, "\rho_{n-1}"] \arrow[d] & \exp \mathcal{P}_{n-1} \rtimes \Sigma_{n-1} \arrow[d] \\ \mathcal{B}_n \arrow[r, "\rho_n"] & \exp \mathcal{P}_n \rtimes \Sigma_n \end{tikzcd}$$

- **Normalization.** The images of the generators $(\sigma_i)_{i=1,n-1}$ satisfy

$$(\mathbf{N}) \qquad \rho_n \sigma_i = u_i \otimes s_i\,, \quad \text{where} \quad u_i \in \exp \mathcal{P}_n \quad \text{and} \quad u_i \equiv_2 1 + \frac{t_{i,i+1}}{2}\,.$$

Remark 2.3. From (E), (Σ) and (N), we obtain the following formula for the pure braid generators (abbreviating from now on $u \otimes \mathrm{id}$ to u):

$$\rho_n \alpha_{ji} \equiv_2 1 + t_{ij}\,.$$

Indeed, $\rho_n \sigma_i^2 = u_i \cdot^{s_i} u_i \equiv_2 (1 + \frac{t_{i,i+1}}{2})^2 \equiv_2 1 + t_{i,i+1}$. Hence, $\rho_n \alpha_{ji} = us \cdot \rho_n \sigma_j^2 \cdot s^{-1} u^{-1} \equiv_2 {}^s\rho_n \sigma_j^2$, where $u \in \exp \mathcal{P}_n$, $s = s_{i-1} \cdots s_{j+1}$ and $\rho_n(\sigma_{i-1} \cdots \sigma_{j+1}) = us$. It is easy to check that ${}^s\rho_n \sigma_j^2 \equiv_2 1 + t_{ji}$, which proves our claim.

3. Drinfel'd associators and Drinfel'd representations

Following Drinfel'd [7], we recall a method for constructing representations having the four properties described in Section 2, based on the notion of *associator*.

3.1. The complete Hopf algebra $\mathbb{K}\langle\langle A, B\rangle\rangle$ has a natural involution s: ${}^sA = B$, ${}^sB = A$. Given $\Phi \in \mathbb{K}\langle\langle A, B\rangle\rangle$, set $\Phi_t := \Phi(t_{12}, t_{23}) \in \mathcal{A}_3$.

Definition 3.1 ([7]). An element $\Phi \in \mathbb{K}\langle\langle A, B\rangle\rangle$ is called an *associator* if it satisfies the following conditions :

$$\begin{array}{ll}
(\mathbf{AE}) & \Phi = \exp(\varphi)\,, \ \text{with } \varphi \in \widehat{\mathbb{L}}^{>1}\langle A, B\rangle \\
(\mathbf{AS}) & {}^s\Phi = \Phi^{-1} \\
(\mathbf{H1}) & \exp(\frac{t_{12}+t_{13}}{2}) = {}^{231}\Phi_t^{-1} \cdot \exp(\frac{t_{13}}{2}) \cdot {}^{213}\Phi_t \cdot \exp(\frac{t_{12}}{2}) \cdot \Phi_t^{-1} \\
(\mathbf{H3}) & \exp(\frac{t_{13}+t_{23}}{2}) = {}^{312}\Phi_t \cdot \exp(\frac{t_{13}}{2}) \cdot {}^{132}\Phi_t^{-1} \cdot \exp(\frac{t_{23}}{2}) \cdot \Phi_t \\
(\mathbf{P}) & \Phi(t_{12}, t_{23} + t_{24})\Phi(t_{13} + t_{23}, t_{34}) = \Phi(t_{23}, t_{34})\Phi(t_{12} + t_{13}, t_{24} + t_{34})\Phi(t_{12}, t_{23})
\end{array}$$

The first two equations must hold in $\mathbb{K}\langle\langle A, B\rangle\rangle$, the next two in $\mathcal{A}_3$, and the last in $\mathcal{A}_4$. Drinfel'd proved that associators exist; see [7, Proposition 5.4].

Definition 3.2. We will say that $\Phi \in \mathbb{K}\langle\langle A, B\rangle\rangle$ is a *semi-associator* if it satisfies the properties from Definition 3.1, except (P).

Let us remark that, in the definition of a semi-associator , (H1) and (H3) are equivalent; see [7, p. 848].

3.2. Drinfel'd [7] used associators to construct families of representations, $\{\rho_n : \mathcal{B}_n \to \mathcal{A}_n \rtimes \mathbb{K}[\Sigma_n]\}$, defined by the formulae:

$$
\begin{cases} \sigma_1 & \mapsto \quad \exp(\frac{t_{12}}{2}) \otimes s_1\,; \\ \sigma_{i>1} & \mapsto \quad \Phi(\sum_{j<i} t_{ji}, t_{i,i+1})^{-1} \cdot (\exp(\frac{t_{i,i+1}}{2}) \otimes s_i) \cdot \Phi(\sum_{j<i} t_{ji}, t_{i,i+1}) \end{cases} \tag{4}
$$

Theorem 3.3 ([7], [22]). *If $\Phi \in \mathbb{K}\langle\langle A, B\rangle\rangle$ is an associator, the above formulae define representations*

$$\rho_n : \mathcal{B}_n \to \mathcal{A}_n \rtimes \mathbb{K}[\Sigma_n]\,,$$

satisfying the properties (E), (Σ), (S), (N) *from Section* 2.

Proof. The only new claim concerns the normalization property. This may be checked as follows. First, it is straightforward to deduce from (4) that $\rho_n\sigma_i = u_i \otimes s_i$, where

$$u_i = \Phi(\sum_{j<i} t_{ji}, t_{i,i+1})^{-1} \cdot \exp(\frac{t_{i,i+1}}{2}) \cdot {}^{s_i}\Phi(\sum_{j<i} t_{ji}, t_{i,i+1})\,. \tag{5}$$

Axiom (AE) implies that $\Phi \equiv_2 1$, which yields property (N). ☐

Another important feature is that all Drinfel'd representations (4) are faithful. This follows from (Σ), [7, p. 848], and [13, Proposition 1.3.3].

4. Universal representations of 3-braids

We may now give the proof of Theorem 1.1.

4.1. Given a representation $\rho : \mathcal{B}_3 \to \mathcal{A}_3 \rtimes \mathbb{K}[\Sigma_3]$, denote by ρ' its restriction to $\mathcal{B}_2$ (embedded in $\mathcal{B}_3$ as explained in §2.1).

Definition 4.1. A representation ρ as above is called *universal* if the family $\{\rho, \rho'\}$ satisfies properties (E), (Σ), (S) and (N) from §2.3.

The set of universal representations of $\mathcal{B}_3$ will be denoted by $\mathcal{U}rep\,(\mathcal{B}_3)$.

Lemma 4.2. *There is a unique representation $\rho' : \mathcal{B}_2 \to \mathcal{A}_2 \rtimes \mathbb{K}[\Sigma_2]$ that satisfies conditions* (E), (Σ) *and* (N), *given by $\rho'\sigma_1 = \exp(\frac{t_{12}}{2}) \otimes s_1$.*

Proof. Properties (E) and (Σ) together are saying that $\rho'\sigma_1 = \exp(\lambda t_{12}) \otimes s_1$, with $\lambda \in \mathbb{K}$. By (N), $\lambda = \frac{1}{2}$. ☐

We parametrize representations $\rho : \mathcal{B}_3 \to \mathcal{A}_3 \rtimes \mathbb{K}[\Sigma_3]$, using another presentation of $\mathcal{B}_3$, derived from (1):

$$\mathcal{B}_3 = \langle\, \sigma_1, \Delta \mid \sigma_1\Delta\sigma_1 = \Delta\sigma_1^{-1}\Delta\,\rangle\,, \tag{6}$$

where $\Delta = \sigma_1\sigma_2\sigma_1$. The fundamental element Δ has the property that its square Δ^2 generates the center of the braid group [5]. We also consider the element of $\mathcal{P}_3$, $T = \frac{1}{2}(t_{12} + t_{13} + t_{23})$ and observe that

$$\mathcal{P}_3 = \mathbb{K} \cdot T \times \widehat{\mathbb{L}}\,\langle\, A := t_{12}, B := t_{23}\,\rangle\,, \tag{7}$$

as Lie algebras. In particular, the center of the Lie algebra $\mathcal{P}_3$ is $\mathbb{K} \cdot T$.

4.2. Let $\Psi = \exp(\psi)$ be a group-like element of $\mathbb{K}\langle\langle A, B\rangle\rangle$. Set

$$\begin{cases} \rho\sigma_1 = \exp(\frac{t_{12}}{2}) \otimes s_1 \in \exp(\mathcal{P}_2) \rtimes \Sigma_2 \\ \rho\Delta = \exp(T) \cdot \Psi_t^{-1} \otimes 321 \in \exp(\mathcal{P}_3) \rtimes \Sigma_3 \end{cases} \tag{8}$$

Theorem 1.1 is a consequence of the following result.

Theorem 4.3. *If Ψ is a semi-associator (in the sense of* Definition 3.2*), then* (8) *defines a universal representation $\rho \in \mathcal{U}rep\,(\mathcal{B}_3)$ (in the sense of* Definition 4.1*). Conversely, every universal representation of $\mathcal{B}_3$ has a unique parametrization of the form* (8)*, where Ψ is a semi-associator .*

We start by noting that Definition 4.1, Lemma 4.2 and presentation (6) readily imply that $\rho \in \mathcal{U}rep\,(\mathcal{B}_3)$ if and only if ρ is of the form (8), with Ψ_t replaced by $\Phi \in \exp(\mathcal{P}_3)$, and the following two properties hold:

$$(\mathbf{YB}) \qquad \rho\Delta = \rho\sigma_2 \cdot \rho\sigma_1 \cdot \rho\sigma_2$$

(expressing the fact that ρ is a representation) and

$$(\mathbf{N2}) \qquad \rho\sigma_2 = u_2 \otimes s_2\,, \quad \text{with} \quad u_2 \equiv_2 1 + \frac{t_{23}}{2}\,.$$

Lemma 4.4. *In the above setting,* (N2) *is equivalent to* $\log \Phi \equiv_2 0$ *in $\mathcal{P}_3$.*

Proof. Since $\Delta = \sigma_1\sigma_2\sigma_1$,

$$\rho\sigma_2 = (\exp(-\frac{t_{12}}{2}) \otimes s_1) \cdot (\exp T \cdot \Phi^{-1} \otimes 321) \cdot (\exp(-\frac{t_{12}}{2}) \otimes s_1)\,.$$

It follows that

$$\rho\sigma_2 = \exp(-\frac{t_{12}}{2}) \cdot \exp T \cdot^{213} \Phi^{-1} \cdot \exp(-\frac{t_{13}}{2}) \otimes s_2\,. \tag{9}$$

Condition (N2) becomes $(1 + \frac{t_{23}}{2}) \cdot^{213} \Phi^{-1} \equiv_2 1 + \frac{t_{23}}{2}$, whence the result. $\square$

By resorting to (7), we infer that $\Phi = \Psi_t$, with $\Psi = \exp(\psi)$ and $\psi \in \widehat{\mathbb{L}}^{>1}\langle A, B\rangle$. The proof of Theorem 4.3 is thus reduced to showing that the Yang–Baxter equation (YB) for Φ is equivalent to properties (AS) and (H) from Definition 3.1 for Ψ. To prove this equivalence, we need a preliminary result.

Lemma 4.5. *Assume $\{\mu_{ij} \in \mathcal{A}_n\}_{1\le i<j\le n}$ are such that $\mu_{ij} \equiv_2 t_{ij}$. Denote by V the $\mathbb{K}$–span of $\{\mu_{ij}\}$. Then, for an arbitrary $t \in \mathcal{A}_n$, we have an expansion $t = \sum_{k\ge 0} v_k$, with the property that $v_k \in V^k := \overbrace{V\cdots V}^{k}$, for $k > 0$, and $v_0 \in V^0 := \mathbb{K}\cdot 1$.*

Proof. By induction on k,

$$0 \equiv_k t - \sum_{i<k} v_i \equiv_{k+1} \sum k - \text{monomials in } (t_{ij}) \equiv_{k+1} \sum k - \text{monomials in } (\mu_{ij})\,,$$

which completes the induction step. □

Next, we use the above lemma to deduce the following.

Lemma 4.6. *Equation* (YB) *for Φ implies condition* (AS) *for Ψ.*

Proof. Knowing that ρ is a representation and $\Delta^2 \in \mathcal{PB}_3$ is central in $\mathcal{B}_3$, we infer that $[\rho\Delta^2, \rho\alpha_{ij}] = 0$ in $\mathcal{A}_3$, for $1 \le i < j \le 3$. We also know from Remark 2.3 that $\rho\alpha_{ij} = 1 + \mu_{ij}$, with $\mu_{ij} \equiv_2 t_{ij}$. Due to Lemma 4.5, we obtain that $\rho\Delta^2$ is central in $\mathcal{A}_3$. Hence, $\rho\Delta^2 = \exp(hT)$, for some $h \in \mathbb{K}$; see (7). On the other hand,

$$\rho\Delta^2 = \rho(\alpha_{13}\alpha_{23}\alpha_{12}) \equiv_2 1 + 2T\,,$$

again by Remark 2.3. This forces $h = 2$. Therefore, $\exp(2T) = (\rho\Delta)^2 = \exp(2T)\cdot \Phi^{-1} \cdot^{321} \Phi^{-1}$, as follows from (8). Finally, since $\Phi = \Psi_t$, ${}^{321}\Phi = \Phi^{-1}$ translates to ${}^{s}\Psi = \Psi^{-1}$, as asserted. □

4.3. Recall that axioms (H1) and (H3) from Definition 3.2 are equivalent. With this remark, the Lemma below will finish the proof of Theorem 4.3.

Lemma 4.7. *The Yang–Baxter equation* (YB) *for Φ is equivalent to the hexagonal axiom* (H3) *for Ψ.*

Proof. First, we may rewrite (9) in the form

$$\rho\sigma_2 = \exp(\frac{t_{13}+t_{23}}{2}) \cdot^{213} \Phi^{-1} \cdot \exp(\frac{-t_{13}}{2}) \otimes s_2\,,$$

since $[t_{12}, t_{13}+t_{23}] = 0$. Together with (8), this leads to

$$\rho\sigma_2\cdot\rho\sigma_1\cdot\rho\sigma_2 = \exp(\frac{t_{13}+t_{23}}{2}) \cdot^{213} \Phi^{-1} \cdot \exp(\frac{t_{12}+t_{23}}{2}) \cdot^{132} \Phi^{-1} \cdot \exp(-\frac{t_{23}}{2}) \otimes 321\,.$$

Comparing this with (8), we find that (YB) is equivalent to

$$\exp(\frac{t_{12}}{2})\cdot\Phi^{-1} =^{213} \Phi^{-1}\cdot \exp(\frac{t_{12}+t_{23}}{2}) \cdot^{132} \Phi^{-1} \cdot \exp(-\frac{t_{23}}{2})\,, \tag{10}$$

or

$$\exp(\frac{t_{12}+t_{23}}{2}) =^{213} \Phi \cdot \exp(\frac{t_{12}}{2}) \cdot \Phi^{-1} \cdot \exp(\frac{t_{23}}{2}) \cdot^{132} \Phi\,. \tag{11}$$

Applying s_2 to (11), we find the equivalent form

$$\exp(\frac{t_{13}+t_{23}}{2}) = {}^{312}\Phi \cdot \exp(\frac{t_{13}}{2}) \cdot {}^{132}\Phi^{-1} \cdot \exp(\frac{t_{23}}{2}) \cdot \Phi\,, \tag{12}$$

which is precisely condition (H3) for Ψ. □

Remark 4.8. It is worth pointing out that the representation ρ corresponding to a semi-associator Ψ, in our Theorem 4.3, is given by (4), as in the Drinfel'd construction of representations coming from an associator.

To check this, start with a group-like element $\Psi \in \mathbb{K}\langle\langle A, B\rangle\rangle$ and set $\Phi = \Psi_t$. For $i = 2$, (4) gives

$$\rho\sigma_2 = \Phi^{-1} \cdot \exp(\frac{t_{23}}{2}) \cdot {}^{132}\Phi \otimes s_2\,. \tag{13}$$

Comparing this with (9), we find that we must verify the equality

$$\Phi^{-1} \cdot \exp(\frac{t_{23}}{2}) \cdot {}^{132}\Phi = \exp(\frac{t_{13}+t_{23}}{2}) \cdot {}^{213}\Phi^{-1} \cdot \exp(-\frac{t_{13}}{2})\,, \tag{14}$$

that is,

$$\exp(\frac{t_{13}+t_{23}}{2}) = \Phi^{-1} \cdot \exp(\frac{t_{23}}{2}) \cdot {}^{132}\Phi \cdot \exp(\frac{t_{13}}{2}) \cdot {}^{213}\Phi\,. \tag{15}$$

By applying s_1, we put (15) in the equivalent form

$$\exp(\frac{t_{13}+t_{23}}{2}) = {}^{213}\Phi^{-1} \cdot \exp(\frac{t_{13}}{2}) \cdot {}^{231}\Phi \cdot \exp(\frac{t_{23}}{2}) \cdot \Phi\,. \tag{16}$$

Finally, (16) above is seen to coincide with axiom (H3) of the semi-associator Ψ, by using the equality ${}^{321}\Phi = \Phi^{-1}$, which corresponds to axiom (AS).

5. Braid-permutation and basis-conjugating groups

We examine analogs of the braid groups and braid algebras from Section 2.

5.1. Denote by $\mathbb{F}_n$ the free group generated by $x_1, \ldots, x_n$. The *braid-permutation group*, $\mathcal{BP}_n \subset \mathrm{Aut}(\mathbb{F}_n)$, was investigated in detail in [9]. Its elements are the automorphisms $a \in \mathrm{Aut}(\mathbb{F}_n)$ acting by

$$a(x_i) = y_i^{-1} x_{s(i)} y_i\,, \quad \text{for} \quad 1 \le i \le n\,, \tag{17}$$

where $y_i \in \mathbb{F}_n$ and $s \in \Sigma_n$, and the group product is composition of automorphisms. The abelianization homomorphism, $\mathrm{Aut}(\mathbb{F}_n) \to \mathrm{Aut}(\mathbb{Z}^n)$, induces a short exact sequence

$$1 \to \mathcal{BC}_n \to \mathcal{BP}_n \to \Sigma_n \to 1 \tag{18}$$

Unlike their braid analogs (2), the above sequences are naturally split by the permutation action of Σ_n on $\{x_1, \ldots, x_n\}$.

The following presentation of $\mathcal{BC}_n$, also known as the *McCool group* of size n, was found in [15]. For $1 \le i \ne j \le n$, denote by a_{ij} the automorphism of $\mathbb{F}_n$ which sends x_i to $x_j^{-1}x_ix_j$ and fixes x_k for $k \ne i$. Then $\mathcal{BC}_n$ is generated by $\{a_{ij}\}_{1\le i\ne j\le n}$, with defining relations

$$\left\{ \begin{array}{lll} (I) & (a_{ik}, a_{jk}) = 1\,, & \forall\ 1 \le i \ne j \ne k \le n\,; \\ (II) & (a_{ij}, a_{ik}a_{jk}) = 1\,, & \forall\ 1 \le i \ne j \ne k \le n\,; \\ (III) & (a_{ij}, a_{kl}) = 1\,, & \forall\ 1 \le i \ne j \ne k \ne l \le n\,, \end{array} \right. \tag{19}$$

where $(x, y) := xyx^{-1}y^{-1}$ stands for the group commutator. It is easy to check that

$$s a_{ij} s^{-1} = a_{s(i),s(j)}\,, \tag{20}$$

for all $1 \le i \ne j \le n$ and $s \in \Sigma_n$. In condensed form, $\mathcal{BP}_n = \mathcal{BC}_n \rtimes \Sigma_n$; see also [6].

By Artin's theorem (see [2, Theorem 1.9 on p.30]), $\mathcal{B}_n$ embeds onto the subgroup of elements in $\mathcal{BP}_n$ fixing $x_1 \cdots x_n$. As noted in [9], the embedding $\mathcal{B}_n \hookrightarrow \mathcal{BP}_n$ is given by

$$\sigma_i = a_{i,i+1}s_i\,, \quad \text{for} \quad 1 \le i < n\,. \tag{21}$$

Clearly, the sequences (2) and (18) are compatible with this embedding. In particular, $\mathcal{PB}_n \hookrightarrow \mathcal{BC}_n$.

We also have a canonical embedding,

$$\mathcal{BP}_{n-1} \hookrightarrow \mathcal{BP}_n\,, \tag{22}$$

defined by $a(x_n) = x_n$, for $a \in \mathcal{BP}_{n-1}$, and commutative diagrams

$$\begin{array}{ccc} \mathcal{B}_{n-1} & \longrightarrow & \mathcal{B}_n \\ \downarrow & & \downarrow \\ \mathcal{BP}_{n-1} & \longrightarrow & \mathcal{BP}_n \end{array} \tag{23}$$

Finally, the projections of braid-permutation groups onto symmetric groups from (18) are compatible with stabilization (22), like in the case of braid groups; see the end of §2.1.

5.2. We describe now the analogs of Artin and braid algebras from §2.2.

Definition 5.1. The *oriented Artin Hopf algebra* is the complete Hopf algebra $\mathcal{O}_n$ obtained from $\mathbb{Q}\langle\langle v_{ij} \mid 1 \le i \ne j \le n\rangle\rangle$ by imposing the relations

$$\left\{ \begin{array}{lll} (I) & [v_{ik}, v_{jk}] = 0\,, & \forall\ 1 \le i \ne j \ne k \le n\,; \\ (II) & [v_{ij}, v_{ik} + v_{jk}] = 0\,, & \forall\ 1 \le i \ne j \ne k \le n\,; \\ (III) & [v_{ij}, v_{kl}] = 0\,, & \forall\ 1 \le i \ne j \ne k \ne l \le n\,. \end{array} \right.$$

The *associated graded* oriented Artin Hopf algebra (with respect to the canonical complete filtration of $\mathcal{O}_n$) is denoted by $\mathcal{O}_n^* = \oplus_{k\ge 0}\mathcal{O}_n^k$. It is a Hopf algebra with

grading, obtained as the quotient of $\mathbb{T}_{\mathbb{Q}}\langle v_{ij}\rangle$, graded by tensor length, by the above relations (I)–(III).

The *oriented Artin Lie algebra* is the complete Lie algebra $\mathcal{L}_n := \mathrm{Prim}(\mathcal{O}_n)$, the quotient of $\widehat{\mathbb{L}}\langle v_{ij}\rangle$ by the relations (I)–(III). The *associated graded* oriented Artin Lie algebra (with respect to the canonical complete filtration of $\mathcal{L}_n$) is denoted by $\mathcal{L}_n^* = \oplus_{k\geq 1}\mathcal{L}_n^k$. It is a Lie algebra with grading, obtained as the quotient of the free $\mathbb{Q}$–Lie algebra $\mathbb{L}\langle v_{ij}\rangle$, graded by bracket length, by the relations (I)–(III).

There is a natural left action of Σ_n on the algebras $\mathcal{O}_n = \mathcal{U}\mathcal{L}_n$ and $\mathcal{L}_n$, defined on generators by ${}^{\pi}v_{ij} = v_{\pi(i),\pi(j)}$, in exponential notation.

Definition 5.2. The *oriented braid algebra* $\mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]$ is the semidirect algebra product, $\mathcal{O}_n \otimes \mathbb{Q}[\Sigma_n]$, with respect to the above Σ_n–action on $\mathcal{O}_n$ (where the twisted multiplication is as in Definition 2.2).

The algebra $\mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]$ contains as a multiplicative subgroup the semidirect group product $\exp\mathcal{L}_n \rtimes \Sigma_n$. We thus have split exact sequences of groups,

$$1 \to \exp\mathcal{L}_n \to \exp\mathcal{L}_n \rtimes \Sigma_n \to \Sigma_n \to 1\,,$$

together with compatible canonical embeddings,

$$\exp\mathcal{L}_{n-1} \rtimes \Sigma_{n-1} \hookrightarrow \exp\mathcal{L}_n \rtimes \Sigma_n\,.$$

5.3. We may now define our analogs of Drinfel'd representations. Send a_{ij} to $\exp(v_{ij}) \in \exp\mathcal{L}_n$, for $1 \leq i \neq j \leq n$. The defining Lie relations of $\mathcal{L}_n$ from Definition 5.1 readily imply that the defining group relations of $\mathcal{BC}_n$ from (19) are respected. Consequently, we obtain a representation,

$$R_n\colon \mathcal{BC}_n \longrightarrow \exp\mathcal{L}_n \subset \mathcal{O}_n\,. \tag{24}$$

It follows from (18) and (20) that

$$R_n \otimes \mathrm{id}\colon \mathcal{BP}_n \longrightarrow \exp\mathcal{L}_n \rtimes \Sigma_n \subset \mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n] \tag{25}$$

is a representation of the braid-permutation group $\mathcal{BP}_n$ into the oriented braid algebra $\mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]$, having the exponential property (E) from §2.3.

5.4. As is well-known, the existence of representations, $\rho_n\colon \mathcal{B}_n \to \mathcal{A}_n \rtimes \mathbb{Q}[\Sigma_n]$, satisfying properties (E), (Σ) and (N) from §2.3, is intimately related to formality properties of ordered configuration spaces of $\mathbb{C}$ and of their fundamental groups, $\mathcal{PB}_n$. To obtain the same formality property for the McCool groups $\mathcal{BC}_n$, we turn to a review of Malcev completion, following [23, Appendix A].

A *Malcev Lie algebra* is a rational Lie algebra E, together with a complete, descending $\mathbb{Q}$-vector space filtration, $\{F_rE\}_{r\geq 1}$, such that:

(1) $F_1E = E$;
(2) $[F_rE, F_sE] \subset F_{r+s}E$, for all r and s;

(3) the associated graded Lie algebra, $\mathrm{gr}_F^*(E) = \bigoplus_{r\geq 1} F_r E/F_{r+1}E$, is generated in degree $*=1$.

For example, the canonical complete filtration of $\mathcal{L}_n$ makes it a Malcev Lie algebra, with $\mathrm{gr}_F^*(\mathcal{L}_n) = \mathcal{L}_n^*$, as Lie algebras; see Definition 5.1.

Let G be a group. The lower central series of G is the sequence of normal subgroups $\{\Gamma_k G\}_{k\geq 1}$, defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1}G = (\Gamma_k G, G)$. Observe that the successive quotients $\Gamma_k G/\Gamma_{k+1}G$ are abelian groups. The direct sum of these quotients, $\mathrm{gr}_\Gamma^*(G) := \oplus_{k\geq 1}\Gamma_k G/\Gamma_{k+1}G$ is the *associated graded Lie algebra* of G. The Lie bracket is induced from the group commutator. Consequently, $\mathrm{gr}_\Gamma^*(G)$ is generated as a Lie algebra by $\mathrm{gr}_\Gamma^1(G)$.

A group homomorphism, $\kappa\colon G \to \exp E$, where E is a Malcev Lie algebra, induces a degree zero morphism of Lie algebras, $\mathrm{gr}^*(\kappa)\colon \mathrm{gr}_\Gamma^*(G)\otimes\mathbb{Q} \to \mathrm{gr}_F^*(E)$. If $\mathrm{gr}^*(\kappa)$ is an isomorphism, κ is called a *Malcev completion* of G. There is a functorial Malcev completion, $\kappa_G\colon G \to \exp E_G$, where E_G is called the Malcev Lie algebra of G. If κ is another Malcev completion, there is an isomorphism of complete Lie algebras, $f : E_G \xrightarrow{\sim} E$, such that $\kappa = \exp(f)\circ\kappa_G$. For example, the Malcev completion of $\mathbb{F}_n$, the free group on $x_1,\dots,x_n$, is given by the tautological representation, $\kappa_n\colon \mathbb{F}_n \to \exp\widehat{\mathbb{L}}\langle x_1,\dots,x_n\rangle$, sending each x_i to $\exp(x_i)$.

Let $G = \langle x_1,\dots,x_n \mid w_1,\dots,w_m\rangle$ be a finitely presented group. Define the Malcev Lie algebra E to be the quotient of $\widehat{\mathbb{L}}\langle x_1,\dots,x_n\rangle$ obtained by imposing the relations $\{\log(\kappa_n(w_j)) = 0\}$. It follows from [18, Theorem 2.2] that the tautological homomorphism,

$$\kappa\colon G \to \exp E\,, \tag{26}$$

is a Malcev completion.

Following D. Sullivan [24], we will say that a finitely presented group G is 1–*formal* if the Malcev Lie algebra E_G is quadratic, that is, obtainable from a free complete Lie algebra $\widehat{\mathbb{L}}\langle x_1,\dots,x_n\rangle$ by imposing homogeneous relations of bracket length two.

The usual approach to the 1-formality property and the construction of a Malcev completion for pure braid groups uses the following ingredients. The starting remark is that $\mathcal{PB}_n = \pi_1 F_n(\mathbb{C})$, where $F_n(\mathbb{C})$ denotes the ordered configuration space of n distinct points in $\mathbb{C}$. Next, note that $F_n(\mathbb{C})$ is the complement to the complex hyperplane arrangement of all diagonals $\{z_i = z_j\}$ in $\mathbb{C}^n$. As such, $F_n(\mathbb{C})$ is a formal space in the sense of D. Sullivan [24], as follows from basic results in arrangement theory, due to Orlik and Solomon [17]. Since the fundamental group of a formal space is 1-formal [24], $\mathcal{PB}_n$ is 1-formal. Moreover, Chen's theory of iterated integrals [4] implies that the monodromy representation of the canonical flat connection on the formal space $F_n(\mathbb{C})$ is a Malcev completion homomorphism

for $\mathcal{PB}_n$. Finally, it turns out that the KZ connection coincides with the canonical flat connection. See also Kohno [12].

We will take a completely different, much simpler, approach for the McCool groups $\mathcal{BC}_n$. More precisely, we will exploit the particularly simple form of their presentations (19). This will enable us to deduce the 1-formality property, and to show that the explicit representations (24) are Malcev completions, using only the definitions.

5.5. It follows from the construction of the functorial Malcev completion κ_G [23, Appendix A] and a result on I–adic filtrations of group rings [4, Proposition 2.2.1] that any Malcev completion κ of a *residually torsion-free nilpotent* group G is faithful, if G is finitely generated. By definition, G has the above residual property if all non-trivial elements of G are detected by homomorphisms $G \to N$, where N is a torsion-free nilpotent group. Consequently, this property is inherited by subgroups.

We are going to derive the faithfulness of our representations (24) from a general result about residual torsion-free nilpotence of Torelli groups. The *Torelli group* T_G of a group G is

$$T_G := \{a \in \mathrm{Aut}(G) \mid a \equiv \mathrm{id} \mod \Gamma_2 G\}\,.$$

It is endowed with the decreasing filtration

$$F_s T_G := \{a \in \mathrm{Aut}(G) \mid a \equiv \mathrm{id} \mod \Gamma_{s+1} G\}$$

($s \geq 1$), which has the property that $\Gamma_s T_G \subset F_s T_G$, for all s.

Proposition 5.3 ([10])**.** *Assume $\cap_k \Gamma_k G = \{1\}$ and $\mathrm{gr}^*_\Gamma(G)$ is torsion-free. Then the Torelli group T_G is residually torsion-free nilpotent.*

Proof. The result is stated by Hain without proof in [10, Section 14]. For the benefit of the reader, we are going to give a proof. By the first assumption on G, any non-trivial element, $\mathrm{id} \neq a \in T_G$, is detected by the natural homomorphism, $T_G \to T_{G_k}$, for some k, where $G_k := G/\Gamma_k G$ is nilpotent and inherits the second hypothesis from G. It will be thus enough to assume that moreover G is nilpotent and to prove that in this case T_G must be torsion-free nilpotent. Nilpotence follows from the obvious fact that $F_{k-1} T_G = \{\mathrm{id}\}$, if $\Gamma_k G = \{1\}$. Torsion-freeness may be verified by induction, as soon as we know that all quotients, $F_s T_G / F_{s+1} T_G$, are torsion-free. By [20, Proposition 2.1], the above Torelli filtration quotient embeds into the degree s derivations of the associated graded Lie algebra $\mathrm{gr}^*_\Gamma(G)$, which has no torsion. □

Note that the groups G from Proposition 5.3 are themselves residually torsion-free nilpotent. The hypotheses of the proposition are satisfied by free groups, see [14]. This implies the residual torsion-free nilpotence of $\mathcal{BC}_n \subset T_{\mathbb{F}_n}$; see (17)

and (18). We may thus recover the well-known residual torsion-free nilpotence of $\mathcal{PB}_n \subset \mathcal{BC}_n$. The proposition also applies to iterated semidirect products of free groups with trivial monodromy action in homology, e. g., fundamental groups of fiber-type arrangements of complex hyperplanes; see [8].

5.6. We are ready for the main result of this section.

Theorem 5.4. *The McCool groups $\mathcal{BC}_n$ have the following properties.*

1. *The group $\mathcal{BC}_n$ is 1-formal.*
2. *The Lie algebra with grading $\mathrm{gr}^*_\Gamma(\mathcal{BC}_n)\otimes\mathbb{Q}$ is isomorphic to the Lie algebra $\mathcal{L}_n^*$ from Definition 5.1.*
3. *The homomorphism R_n from (24) is a Malcev completion.*
4. *The above representation R_n is faithful.*

Proof. Part (1). By (26) and (19), the Malcev Lie algebra of $\mathcal{BC}_n$ is isomorphic to the quotient of $\widehat{\mathbb{L}}\langle v_{ij} \mid 1 \le i \ne j \le n\rangle$ by the relations

$$\begin{cases} (I) & \log((\exp(v_{ik}), \exp(v_{jk})))\,, & \forall\, 1 \le i \ne j \ne k \le n\,;\\ (II) & \log((\exp(v_{ij}), \exp(v_{ik})\cdot\exp(v_{jk})))\,, & \forall\, 1 \le i \ne j \ne k \le n\,;\\ (III) & \log((\exp(v_{ij}), \exp(v_{kl})))\,, & \forall\, 1 \le i \ne j \ne k \ne l \le n\,. \end{cases}$$

Using [19, Lemma 2.5], the above relations may be replaced by

$$\begin{cases} (I') & [v_{ik}, v_{jk}]\,, & \forall\, 1 \le i \ne j \ne k \le n\,;\\ (II') & [v_{ij}, \log(\exp(v_{ik})\cdot\exp(v_{jk}))]\,, & \forall\, 1 \le i \ne j \ne k \le n\,;\\ (III') & [v_{ij}, v_{kl}]\,, & \forall\, 1 \le i \ne j \ne k \ne l \le n\,. \end{cases}$$

Due to (I'), (II') becomes

$$(II'') \quad [v_{ij}, v_{ik} + v_{jk}]\,.$$

Since all relations (I'), (II'') and (III') are quadratic, we are done.

Part (2). We have seen that $E_{\mathcal{BC}_n} \cong \mathcal{L}_n$; see Definition 5.1. By the defining property of the Malcev completion of a group (see §5.4), $\mathrm{gr}^*_\Gamma(\mathcal{BC}_n)\otimes\mathbb{Q} \cong \mathrm{gr}^*_F(\mathcal{L}_n) = \mathcal{L}_n^*$.

Part (3). It is enough to verify that $\mathrm{gr}^*(R_n) = \mathrm{id}\colon \mathrm{gr}^*_\Gamma(\mathcal{BC}_n)\otimes\mathbb{Q} \to \mathcal{L}_n^*$. Both Lie algebras being generated in degree one, it suffices to check this on generators, i. e., to show that $R_n(a_{ij}) \equiv_2 1 + v_{ij}$, for all $i \ne j$, which is clear from the definition of R_n.

Part (4). Follows from the residual torsion-free nilpotence of $\mathcal{BC}_n$; see Proposition 5.3 and the discussion of Torelli groups of free groups. □

Remark 5.5. Unlike residual properties, 1-formality is not necessarily inherited by subgroups. For this reason, it seems hopeless to deduce from Theorem 5.4 (1) an alternative simple proof for 1-formality of pure braid groups.

Remark 5.6. The *upper-triangular* McCool groups, $\mathcal{BC}_n^+ \subset \mathcal{BC}_n$, were examined in detail in [6]. It follows from [6, Sections 4–5] that $\mathcal{BC}_n^+$ is generated by $\{a_{ij} \mid n \geq i > j \geq 1\}$, with the following sublist of (19) as defining relations:

$$(I) \quad \text{with} \quad i, j > k\,; \quad (II) \quad \text{with} \quad i > j > k\,; \quad (III) \quad \text{with} \quad i > j, k > l\,.$$

The proof of Theorem 5.4 applies verbatim to these groups, and gives the following information. The group $\mathcal{BC}_n^+$ is 1-formal. The Lie algebra with grading $\mathrm{gr}^*_\Gamma(\mathcal{BC}_n^+) \otimes \mathbb{Q}$ is generated by $\{v_{ij} \mid n \geq i > j \geq 1\}$, with defining relations as in Definition 5.1, subject to the restrictions on indices described above. (Note that the authors of [6] obtain the same presentation for $\mathrm{gr}^*_\Gamma(\mathcal{BC}_n^+)$, over $\mathbb{Z}$, by using a different method.) A Malcev completion for $\mathcal{BC}_n^+$, R_n^+, may be obtained by completing $\mathrm{gr}^*_\Gamma(\mathcal{BC}_n^+) \otimes \mathbb{Q}$ with respect to the degree filtration, and then defining $R_n^+(a_{ij}) = \exp(v_{ij})$, for $i > j$. The representation R_n^+ is faithful.

6. New universal representations and finite type invariants

We will show that the family of representations, $\{R_n \otimes \mathrm{id} \colon \mathcal{BP}_n \to \mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]\}$, constructed in §5.3, shares the essential properties of a Drinfel'd family, $\{\rho_n \colon \mathcal{B}_n \to \mathcal{A}_n \rtimes \mathbb{K}[\Sigma_n]\}$, listed in §3.2. This leads to a geometric interpretation of the family $\{R_n \otimes \mathrm{id}\}$, in terms of finite type invariants for welded braids.

6.1. Denote by $\rho'_n \colon \mathcal{B}_n \to \mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]$ the restriction of $R_n \otimes \mathrm{id}$ to $\mathcal{B}_n$. Recall from §§5.1–5.2 that one has projections, $\mathcal{BP}_n \twoheadrightarrow \Sigma_n$ and $\exp \mathcal{L}_n \rtimes \Sigma_n \twoheadrightarrow \Sigma_n$. One also has inclusions, $\mathcal{BP}_{n-1} \hookrightarrow \mathcal{BP}_n$ and $\exp \mathcal{L}_{n-1} \rtimes \Sigma_{n-1} \hookrightarrow \exp \mathcal{L}_n \rtimes \Sigma_n$. So, it makes sense to speak about properties (E), (Σ) and (S) from §2.3, for $\{R_n \otimes \mathrm{id}\}$ and $\{\rho'_n\}$.

Theorem 6.1. *Both families, $\{R_n \otimes \mathrm{id}\}$ and $\{\rho'_n\}$, have the following properties.*

1. *They satisfy conditions* (E), (Σ) *and* (S).
2. *They consist of faithful representations.*
3. *They are normalized by: $R_n \otimes \mathrm{id}(a_{ij}) \equiv_2 1 + v_{ij}$, for $1 \leq i \neq j \leq n$, respectively $\rho'_n \sigma_i = u_i \otimes s_i$, where $u_i \in \exp \mathcal{L}_n$ and $u_i \equiv_2 1 + v_{i,i+1}$, for $1 \leq i < n$.*

Proof. Part (1). Property (E) was noticed at the end of §5.3. The other two conditions are direct consequences of (25), via our discussion of symmetry and stability morphisms from §§5.1–5.2.

Part (2). Follows from Theorem 5.4(4).

Part (3). Use the definition of $R_n \otimes \mathrm{id}$ and ρ'_n, together with (21). $\square$

Thus, Theorem 1.2 from the Introduction is established.

6.2. The *finite type* invariants for classical braids and links are defined by conditions coming from the (iterated) application of a local move: exchanging negative and positive crossings, in the associated diagram of a projection.

The groups $\mathcal{BP}_n$ were given a geometric interpretation in [9]. They may be identified with *welded braid groups*, consisting of braids that may also have singular points (welds), modulo certain allowable moves. The natural local move in this context is to replace a weld by a positive crossing.

More formally, let $J \subset \mathbb{Q}[\mathcal{BP}_n]$ be the two-sided ideal generated by $\{\sigma_i - s_i\}_{1 \leq i < n}$. The J–adic filtration $\{J^k\}_{k \geq 0}$ will then play the role of the *Vassiliev filtration* of classical braids. Let $\{F_k\mathcal{O}_n\}_{k \geq 0}$ be the canonical complete filtration of $\mathcal{O}_n$. Define on $\mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]$ the multiplicative filtration $F_k := F_k\mathcal{O}_n \otimes \mathbb{Q}[\Sigma_n]$, having the property that $F_k \cdot F_l \subset F_{k+l}$, for all k, l. View $R_n \otimes \mathrm{id}$ as an algebra map, $R_n \otimes \mathrm{id} \colon \mathbb{Q}[\mathcal{BP}_n] \to \mathcal{O}_n \rtimes \mathbb{Q}[\Sigma_n]$. Our next result establishes that $R_n \otimes \mathrm{id}$ is a *universal invariant à la Vassiliev*, for welded braids.

A similar result is known for braids. Let $V \subseteq \mathbb{Q}[\mathcal{B}_n]$ be the two-sided ideal generated by $\{\sigma_i - \sigma_i^{-1}\}_{1 \leq i < n}$. Consider the V-adic filtration $\{V^k\}_{k \geq 0}$ on $\mathbb{Q}[\mathcal{B}_n]$, and the multiplicative filtration $\{F_k\mathcal{A}_n \otimes \mathbb{Q}[\Sigma_n]\}_{k \geq 0}$ on $\mathcal{A}_n \rtimes \mathbb{Q}[\Sigma_n]$, where $\{F_k\mathcal{A}_n\}$ is the canonical complete filtration of $\mathcal{A}_n$. Extend the Drinfel'd representation ρ_n to an algebra map, $\rho_n \colon \mathbb{Q}[\mathcal{B}_n] \to \mathcal{A}_n \rtimes \mathbb{Q}[\Sigma_n]$. Then ρ_n respects filtrations and induces a degree zero multiplicative isomorphism at the associated graded level.

Theorem 6.2. *The algebra map $R_n \otimes \mathrm{id}$ respects the above filtrations and induces a degree zero multiplicative isomorphism at the associated graded level,*

$$\mathrm{gr}^*(R_n \otimes \mathrm{id}) \colon \mathrm{gr}^*_J(\mathcal{BP}_n) \xrightarrow{\sim} \mathcal{O}^*_n \rtimes \mathbb{Q}[\Sigma_n]\,.$$

Thus, the dimensions of the vector spaces dual to $\mathcal{O}^k_n \otimes \mathbb{Q}[\Sigma_n]$ (respectively $\mathcal{A}^k_n \otimes \mathbb{Q}[\Sigma_n]$) may be interpreted as the maximal number of linearly independent finite type weight systems for welded braids (respectively braids) of degree k, for all k. These numbers are known for braids, since Kohno computed in [12] the Hilbert series of $\mathcal{A}^*_n$. In the case of $\mathcal{O}^*_n$, the determination of the Hilbert series seems more difficult than for $\mathcal{A}^*_n$.

The strategy of proving a similar result for $\mathcal{B}_n$, see [21, Theorem 1.1], may be adapted to deduce the above theorem (actually, things are here easier than in [21], since $R_n \otimes \mathrm{id}$ is a multiplicative map). The starting point is the following.

Lemma 6.3. *Set $B = \mathbb{Q}[\mathcal{BP}_n]$ and let I be the augmentation ideal of $\mathcal{BC}_n$. Then $J^k = BI^kB = BI^k = I^kB$, for all k.*

Proof. For the last two equalities, follow the proof of Lemma 2.1 from [21]. It remains to show that $J = BIB$. Since plainly BIB is the kernel of the algebra map $\mathbb{Q}[\mathcal{BP}_n] \twoheadrightarrow \mathbb{Q}[\Sigma_n]$ (see (18)), we infer that $J \subset BIB$. To obtain the other inclusion, it is enough to check that $a_{ij} \equiv 1 \bmod J$, for $1 \leq i \neq j \leq n$. Since

$a_{i,i+1} = \sigma_i s_i^{-1}$, see (21), we may pick $s \in \Sigma_n$ fixing i and sending $i+1$ to j, to deduce from (20) that $a_{ij} = s(\sigma_i s_i^{-1})s^{-1} \equiv 1 \bmod J$. □

6.3. *Proof of Theorem* 6.2. Checking that $R_n \otimes \mathrm{id}(J^k) \subset F_k$, for all k, amounts to verifying that $R_n \otimes \mathrm{id}(\sigma_i - s_i) \in F_1 \mathcal{O}_n \otimes \mathbb{Q}[\Sigma_n]$, for $1 \leq i < n$. By the definition of R_n and (21), $R_n \otimes \mathrm{id}(\sigma_i - s_i) = (\exp(v_{i,i+1}) - 1) \otimes s_i$, which proves Theorem 6.2, except for the fact that each map $\mathrm{gr}^k(R_n \otimes \mathrm{id})$ is a $\mathbb{Q}$-linear isomorphism.

To finish the proof, we introduce an intermediate object, namely the graded vector space $\mathrm{gr}^*_I(\mathcal{BC}_n) := \oplus_{k\geq 0} I^k/I^{k+1}$. Since $R_n : \mathcal{BC}_n \to \exp \mathcal{L}_n$ is a Malcev completion, by Theorem 5.4(3), the general theory from [23] guarantees the fact that the induced map, $\mathrm{gr}^*(R_n)\colon \mathrm{gr}^*_I(\mathcal{BC}_n) \to \mathrm{gr}^*_F(\mathcal{UL}_n = \mathcal{O}_n)$ is an isomorphism. Hence, we obtain a degree zero isomorphism,

$$\mathrm{gr}^*(R_n) \otimes \mathrm{id}\colon \mathrm{gr}^*_I(\mathcal{BC}_n) \otimes \mathbb{Q}[\Sigma_n] \xrightarrow{\sim} \mathcal{O}^*_n \otimes \mathbb{Q}[\Sigma_n]\,.$$

The exact sequence (18) gives rise to a vector space isomorphism, $\Psi\colon \mathbb{Q}[\mathcal{BC}_n] \otimes \mathbb{Q}[\Sigma_n] \xrightarrow{\sim} \mathbb{Q}[\mathcal{BP}_n]$, defined by $\Psi(c \otimes s) = c \cdot s$, for $c \in \mathcal{BC}_n$ and $s \in \Sigma_n$. Due to Lemma 6.3, the argument from [21, §2.2] shows that Ψ identifies the filtrations $\{I^k \otimes \mathbb{Q}[\Sigma_n]\}$ and $\{J^k\}$. Consequently, we obtain another degree zero isomorphism,

$$\mathrm{gr}^*(\Psi)\colon \mathrm{gr}^*_I(\mathcal{BC}_n) \otimes \mathbb{Q}[\Sigma_n] \xrightarrow{\sim} \mathrm{gr}^*_J(\mathcal{BP}_n)\,.$$

We finish by showing that $\mathrm{gr}^*(R_n \otimes \mathrm{id}) \circ \mathrm{gr}^*(\Psi) = \mathrm{gr}^*(R_n) \otimes \mathrm{id}$, which follows at once from the definition of $R_n \otimes \mathrm{id}$ and Ψ. □

This completes the proof of Theorem 1.3 from the Introduction.

Since plainly $V^k \subseteq J^k$, for all k, we infer that the representations ρ'_n (obtained by restricting $R_n \otimes \mathrm{id}$ to $\mathbb{Q}[\mathcal{B}_n]$) give rise to finite type invariants for braids. At the associated graded level, the graded algebra map $\mathrm{gr}^*_V(\mathcal{B}_n) \to \mathrm{gr}^*_J(\mathcal{BP}_n)$, induced by the inclusion $\mathcal{B}_n \hookrightarrow \mathcal{BP}_n$, may be identified, via the isomorphisms provided by ρ_n and $R_n \otimes \mathrm{id}$, with $\delta_n \otimes \mathrm{id} : \mathcal{A}^*_n \rtimes \mathbb{Q}[\Sigma_n] \to \mathcal{O}^*_n \rtimes \mathbb{Q}[\Sigma_n]$. Here, the Hopf algebra map $\delta_n : \mathcal{A}^*_n \to \mathcal{O}^*_n$ sends the generator t_{ij} to $v_{ij} + v_{ji}$, for any $1 \leq i \neq j \leq n$.

The injectivity of δ_n is equivalent to the fact that all finite type invariants for n-braids come from welded braids. We have checked that this holds for $n \leq 3$. It would be interesting to know what happens in general.

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